1. Consider the cross-sectional view of a planar waveguide structure based on Silicon guiding layer and Silicon Dioxide cladding layers as shown in the figure. Calculate the following quantities at an operating $\lambda_0 = 1.55 \, \mu m$:

(a) V number
(b) Number of even and odd TE modes that will be guided in this structure
(c) Effective index ($n_{\text{eff}} = \beta/k_0$) of each of these modes
(d) Calculate the value to which the thickness, $d$, of the guiding layer needs to be reduced to make it a single mode waveguide.

2. Use the effective index method to find the “net” effective index in the below waveguide. Assume $n_1 = 3.55$ (for InGaAsP), $n_0 = 3.17$ (for InP), $W = 2 \, \mu m$ and $T = 0.2 \, \mu m$.

Hints: You do not need to calculate the mode profiles, just find the final (net) effective index. Also, reading numbers (roughly) from Fig. 2 of the attached classical paper, will simplify your calculations a lot.
Scaling Rules for Thin-Film Optical Waveguides

H. Kogelnik and V. Ramaswamy

An asymmetry measure is introduced to characterize thin-film optical waveguides that are asymmetric in refractive index. Together with the usual normalized frequency this allows the plotting of universal charts from which the guide cutoff, the effective guide index, and the effective guide thickness can be determined by the use of simple scaling rules. The minimum value of the effective guide thickness is found to be a simple function of wavelength and the film and substrate indices.

I. Introduction

The thin-film optical waveguides employed in the guided-wave devices of integrated optics\textsuperscript{1-3} consist usually of three different materials, one for the substrate, one for the film, and the third for the cover or superstrate. Even if the guide can be considered as an infinitely extending slab or sheet, the designer interested in such guide characteristics as the effective guide index and the effective guide width is confronted with five parameters. These are the refractive indices $n_s$, $n_f$, and $n_c$ of the substrate, the film, and the cover, the film-thickness $f$, and the free space wavelength $\lambda$ or propagation constant $k = 2\pi/\lambda = \omega/c$ of the light. The purpose of this paper is to show how the number of independent parameters can be reduced by the introduction of appropriately normalized parameters and associated scaling rules and to provide universal plots from which the effective guide index and the effective guide thickness can be determined for any slab-guide configuration. Together with the scaling rules these plots give a compact overview of the above waveguide characteristics and of the various design possibilities.

Clearly, one basic guide parameter to use is the normalized frequency or film thickness $V$, defined as

$$V = k f (n_f^2 - n_s^2)^{1/2}. \quad (1)$$

Similar normalizations have already been employed with advantage in the analyses of optical fibers\textsuperscript{4,5} and of slab\textsuperscript{6} and rectangular\textsuperscript{7} film guides.

As a second basic parameter we introduce a measure for the index asymmetry of the waveguide structure. This measure is defined in a somewhat different way for the TE and the TM modes, and its values for practical guide configurations can range from zero (for perfect symmetry) to infinity.

For TE modes these two basic parameters are sufficient to allow the charting of universal plots. This turns out to be possible even for cases where the differences between the substrate and film indices are large. For TM modes an additional parameter is necessary in the general case. But if the differences between the substrate and film indices are small (while allowing arbitrary indices for the cover!) the number of required parameters reduces to the two upper ones, and one finds that the universal plots of the TE-modes can also be used for the TM-modes. This is particularly gratifying because the difference between the substrate and film indices is, indeed, small in the majority of practical cases. We propose to investigate the universal plots in some detail. In addition, we will also explore the range of their applicability to TM modes as the differences between the substrate and film indices become larger.

We consider asymmetric slab waveguides such as that sketched in Fig. 1. The propagation constant $\beta$ of the guide and the related effective guide index

$$N = \beta/k \quad (2)$$

are determined by the dispersion relation\textsuperscript{3}

$$\kappa f = m \pi + \phi_s + \phi_c \quad m = 0, 1, 2, \ldots, \quad (3)$$

which is the basis for our discussion. Here $m$ is the mode number, $\phi_s$ and $\phi_c$ are the phase shifts associated with total reflection of the light from the film-substrate and film-cover interfaces, and $\kappa$ is the transverse propagation constant given by

$$k^2 = k^2(n_f^2 - N^2). \quad (4)$$

To determine the phase shifts and the effective guide width we also need the decay constants $\gamma_s$ and $\gamma_c$ in substrate and cover, which are given by\textsuperscript{3,8}

$$\gamma_s^2 = k^2(N^2 - n_s^2)$$
$$\gamma_c^2 = k^2(N^2 - n_c^2). \quad (5)$$

After these general remarks we shall deal separately with the TE modes and the TM modes.

The authors are with Bell Laboratories, Crawford Hill Laboratory, Holmdel, New Jersey 07733.

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II. TE Modes: Guide Index and Effective Width

For TE modes it is convenient to use a parameter \( b \) which we call the normalized guide index and which is defined by

\[
N^2 = n_s^2 + b(n_f^2 - n_s^2).
\]

(6)

Assuming \( n_s \geq n_c \) this normalized index takes on values between zero and unity. At the cutoff of the guide we have \( N = n_s \) and \( b = 0 \), and very far away from cutoff we have \( N = n_f \) and \( b = 1 \). In addition to this convenient range there is the simple linear relation

\[
N = n_s + b(n_f - n_s),
\]

(7)

which holds for small differences of the substrate and film indices \([n_f - n_s]/n_f < 1\).

In the case of TE modes, the formulas for the phase shifts \( \phi_s \) and \( \phi_c \) are

\[
\tan \phi_s = \gamma_s/k,
\]

\[
\tan \phi_c = \gamma_c/k.
\]

(8)

We use these formulas together with Eqs. (4) and (5), and the definitions for \( V \) and \( b \), to rewrite the dispersion relation Eq. (3) in the normalized form

\[
V(1 - b)^{1/2} = m \pi + \tan^{-1} [b/(1 - b)]^{1/2} + \tan^{-1} [(b + a)/(1 - b)]^{1/2}.
\]

(9)

This form indicates that the guide index \( b = b(V,a) \) depends on only two independent parameters: the normalized frequency \( V \) and the asymmetry measure \( a \) defined by

\[
a = (n_f^2 - n_s^2)/(n_f^2 - n_c^2).
\]

(10)

This measure is related to the \( \eta \) parameter used by Anderson.\(^6\) As we assume that \( n_f > n_s > n_c \), the measure \( a \) can range in value from zero for perfect symmetry \( (a = 0 \text{ if } n_s = n_c) \) to infinity for strong asymmetry \((a \to \infty \text{ if } n_s \neq n_c \text{ and } n_f \to n_f)\). To illustrate this, Table I lists the refractive indices of three practical guide structures together with the corresponding asymmetry measure \((a = a_E)\). The first example shows the waveguide parameters\(^9\) of a GaAlAs heterostructure laser at 0.9 \( \mu \)m. The second example refers to a sputtered Corning 7059 glass waveguide\(^4\) at 0.6328 \( \mu \)m and the third to the outdiffused guide\(^10\) of a LiNbO\(_3\) modulator at 0.6328 \( \mu \)m (where the relevant extraordinary indices are listed).

For \( b = 0 \) and \( m = 0 \), Eq. (9) yields the normalized cutoff frequency \( V_0 \) for the fundamental mode as

\[
V_0 = \tan^{-1}(a)^{1/2},
\]

which, in terms of film thickness \( f \) and wavelength \( \lambda \), takes on the form

\[
(f/\lambda)_0 = [1/2\pi(n_f^2 - n_s^2)^{1/2}] \tan^{-1}(a)^{1/2}.
\]

(12)

This reflects the well-known fact that there is no cutoff for a symmetrical guide \((V = 0 \text{ if } a = 0)\). For a highly asymmetrical guide \((a \to \infty)\) such as the LiNbO\(_3\) guide listed above we get a cutoff at \( V_0 = \pi/2 \), or

\[
(f/\lambda)_0 = (\pi/2)(n_f^2 - n_s^2)^{-1/2}.
\]

(13)

The cutoff \( V_m \) for the mode of \( m \)th order occurs at

\[
\begin{array}{c|cccc}
\text{Waveguide} & n_s & n_f & n_c & a_E & a_M \\
\hline
\text{GaAlAs, double heterostructure} & 3.55 & 3.6 & 3.55 & 0 & 0 \\
\text{Sputtered glass} & 1.515 & 1.62 & 1 & 3.9 & 27.1 \\
\text{Outdiffused LiNbO}_3 & 2.214 & 2.215 & 1 & 881 & 21,206 \\
\end{array}
\]

Fig. 2. Guide index \( b \) as a function of normalized frequency \( V \) for the lowest three TE-mode orders for various degrees of asymmetry.
Fig. 3. Guide index $b$ for the single-TE-mode regime. Cutoff values are indicated by circles.

$$V_m = V_0 + m \pi. \quad (14)$$

When the mode number $m$ is large, $V_0 \ll m \pi$, and the above expression becomes the well-known formula for the number of allowed guided modes

$$m = \frac{2f/\lambda}{(n_f^2 - n_o^2)^{1/2}}. \quad (15)$$

Our first universal chart is a plot of Eq. (9) showing the dependence of the guide index $b$ on the normalized frequency $V$ for various values of the measure $a$. Figure 2 shows these dispersion curves for the first three modes. Figure 3 shows the same curves on an expanded scale for the fundamental and additional values for $a$. On this figure we have also marked the single-mode limit $V_1$ (i.e., the first order cutoff).

To discuss the nature of these dispersion curves in more detail we calculate the slope from Eq. (9), which is

$$\frac{db}{dV} = \frac{2(1-b)}{V + \frac{1}{(b+a)^{1/2}} + \frac{1}{(b+a)^{1/2}}} = \frac{2(1-b)}{W}. \quad (16)$$

As indicated by the right-hand side of this equation, we find that the slope is related to the effective width of the guide defined later in Eq. (22). From Eq. (16) it follows that all curves have zero slope at cutoff,

$$\frac{db}{dV} \bigg|_{b=0} = 0. \quad (17)$$

Very far from cutoff ($b = 1$), one can use the expansion

$$\tan^{-1} x = (\pi/2) - (1/x) + \ldots \quad \text{for } x \gg 1 \quad (18)$$

in Eq. (9) and obtains the approximate formula

$$b = 1 - (m + 1)\pi^2/[W + 1 + [1/(1+a)]^{1/2}], \quad (19)$$

which corresponds to the asymptotic solutions that have been found to give highly satisfactory predictions in several cases.\textsuperscript{7,8,11}

[There is no Eq. (20)]

Our second quantity of interest is the effective guide thickness $w$, which is a measure for the region in which the guided light is concentrated and which is defined by\textsuperscript{3,12}

$$w = f + (1/\gamma_c) + (1/\gamma_e). \quad (21)$$

We use a normalization analogous to that for $V$ and define a normalized guide thickness $W$ by

$$W = kw(n_f^2 - n_o^2)^{1/2}. \quad (22)$$

This, together with Eqs. (5) and (6), allows us to rewrite Eq. (21) in the normalized form

$$W = V + [1/(b^{1/2}) + [1/(b+a)]^{1/2}. \quad (23)$$

As $b = b(V,a)$ we have a normalized guide thickness $W = W(V,a)$ which, like $b$, is a function of only two parameters. Figure 4, our second universal plot, shows $W$ as a function of $V$ for various values of the measure $a$ for the fundamental mode. As expected, $W$ increases to infinity at cutoff. When $V$ is large, the guide thickness $W$ increases linearly with $V$ following the asymptotic expression

$$W = V + 1 + 1/(1+a)^{1/2}, \quad (24)$$

We notice that the right-hand side of this expression appears as a term in Eq. (19). The asymptotes for $a = 0$ ($W = V + 2$) and $a = \infty$ ($W = V + 1$) are shown in the figure as dashed lines.

The envelope of the $W(V)$ curves shows a quite broad minimum, with the smallest minimum of

$$W_{\text{min}} = 4.40 \quad (25)$$

occurring at $V = 2.55$ for $a = \infty$, and which does not change appreciably until $a = 1$. The minimum for

![Fig. 4. Normalized effective guide thickness $W$ as a function of normalized frequency $V$ for the fundamental TE mode.](image-url)
symmetric guides \((a = 0)\) is somewhat larger (4.93 at \(V = 1.73\)). Equation (25) predicts a minimum value for the effective guide width \(w_{min}\)
\[
w_{min}/\lambda = 0.7/(n_s^2 - n_i^2)^{1/2},
\]
which depends only on the wavelength and the film and substrate indices.

III. TM Modes: Guide Index and Effective Width

The situation for the TM modes is analogous, but not quite as simple as for the TE modes. The formulas for the phase shifts appearing in the dispersion equation are
\[
\tan \phi_s = (n_i^2/n_s^2)(\gamma_s/\kappa) \\
\tan \phi_s = (n_i^2/n_s^2)(\gamma_s/\kappa).
\]

The somewhat different form of these expressions compared to those for the TE modes makes it necessary to define the normalized guide index \(b\) in a different way, namely,
\[
b = [(N^2 - n_s^2)/(n_i^2 - n_s^2)][n_i^2/(n_i^2 q_s)],
\]
which, after some algebraic manipulation, is seen to be equivalent to
\[
N^2 = [n_i^2(1-b) + n_i^2 b]q_s
\]
and where the reduction factor \(q_s\) is given by
\[
q_s = N^2/n_i^2 + N^2/n_s^2 - 1 = n_s/n_i^2 (1-b) + bn_i^2/n_s^2.
\]

This is a somewhat more complicated definition, but it maintains the range of \(b = 0\) at guide cutoff and \(b = 1\) far away from cutoff. The simple relation of Eq. (7) for small film-substrate index differences also remains intact. With these definitions, the normalized form of the dispersion relation for the TM modes becomes
\[
\frac{V(q_s)^{1/2}n_s/n_s(1-b)\sqrt{b}}{1-b} = m\pi \\
+ \tan^{-1} \left( \frac{b}{1-b} \right)^{1/2} + \tan^{-1} \left[ \frac{b + a(1-b)}{1-b} \right]^{1/2}.
\]

Here the asymmetry measure for TM modes has been defined as
\[
a = (n_i^4/n_s^4)[(n_s^2 - n_i^2)/(n_i^2 - n_s^2)]
\]
and
\[
d = (1 - n_i^2/n_s^2)(1 - n_i^2/n_s^2).
\]

Note that we have defined the asymmetry measure in a somewhat different way for the TM modes than for the TE modes, and the dispersion Eq. (30) has a more complicated form. Apart from \(V\) there are now at least two independent parameters needed to represent a general waveguide structure. As each one of the parameters \(a, d,\) and \(q\) depends only on the index ratios \(n_s/n_i\) and \(n_i/n_f\) [see Eqs. (29), (31), and (32)], these two ratios can be used as independent parameters. However, we find it more convenient to use the parameters \(a\) and \(n_s/n_i\) instead.

The above choice of parameters allows various useful simplifications, in particular when the film-substrate index difference is small \((n_s/n_f \approx 1)\). In this case we have the approximation for the term
\[
(n_i^2/n_s^2)q_s = 1 + 2b(1 - n_s^2/n_f^2),
\]
and can set \((q_s)^{1/2}n_f/n_s = 1 = d\) in the dispersion Eq. (30). As a result the latter assumes a form that is exactly the same as that of the corresponding Eq. (9) for the TE modes. This means that we can use the universal dispersion chart of Figs. 2 and 3 for the TM modes also, with the only difference between the TE and TM modes being the different definitions and values of the asymmetry measure \(a\). In Table I we have included the values of \(a = a_M\) for the TM modes for our waveguide examples, and we note considerable differences between \(a_S\) and \(a_M\).

Let us, now, explore what happens when \((n_f - n_s)\) gets larger and consider first the dispersion relation at and near cutoff. We note that for \(b = 0\) the terms \((q_s)^{1/2}n_f/n_s\) and \(d\) drop out for any value of \((n_f - n_s)\), and we have exactly the same relations Eqs. (11)-(14) for the cutoff frequencies \(V_0\) and \(V_m\) as in the case of the TE modes. Next consider the term \(bd\). Clearly it is unimportant if we have almost symmetrical guides \((a = 0)\); and a detailed numerical investigation shows that this term can be neglected for just about every practical case: We have computed dispersion curves from Eq. (30) for a series of values of \(a\) in the interval \(a = 0\) and \(a = \infty\) and various ratios of \(n_s/n_f\) between 1 and 0.7 for the frequency range \(0 \leq V \leq 10\) of the fundamental mode. The example of \(a = 1\) is shown in Fig. 5. In all considered cases, it turned out that neglecting the term \(bd\) results in errors in \(b\) that are smaller than 1%.

![Fig. 5. Guide index \(b\) as a function of normalized frequency \(V\) for the fundamental TM mode with asymmetry \(a = a_M = 1\) and \(n_s/n_f = 0.7 - 1\).](image-url)
Neglecting the term \(bd\) makes the right-hand sides of Eqs. (9) and (30) identical and leads to the relation

\[
V_E(b, a) = V_M(b, a)[(q_s)^2 n_f/n_s]
\]  

(34)

between the normalized frequencies \(V_E\) and \(V_M\) of the TE and TM modes. This allows us to adapt the chart of Figs. 2 and 3 for use with the TM modes even when \((n_f - n_s)\) is large: For given \(b\) we can read off the value of \(V_E\) and calculate \(V_M\) from the above relation. This leads to the corrected values shown in Fig. 5 for \(a = 1\). No corrections are necessary when \(n_s/n_f \approx 1\) as indicated before. For \(n_s/n_f = 0.9\), we see that the corrections lead to changes of up to 10% in the \(b\) values. These increase to almost 15% when \(a \to \infty\).

Just as in the TE case, we can derive an asymptotic formula for the TM modes. We obtain it by expanding Eq. (30) for large \(V\) (small \(1 - b\)) with the result

\[
b = 1 - (m + 1)^2 \pi^2/[V(n_f^2/n_s^2) + 1 + 1/(1 + a - ad)^{1/2}]^2
\]

(35)

for \(V \gg 1\), where the term under the square root can be written in the form

\[
\sqrt{V(n_f^2/n_s^2) + 1 + 1/(1 + a - ad)^{1/2}}
\]
Table II. Waveguides Parameters

<table>
<thead>
<tr>
<th>Mode</th>
<th>TE</th>
<th>TM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normalized frequency</td>
<td>$V = k_f (n_r^2 - n_s^2)^{1/2}$</td>
<td>$W = k_w (n_r^2 - n_s^2)^{1/2}$</td>
</tr>
<tr>
<td>Normalized effective thickness</td>
<td>$W = k_w (n_r^2 - n_s^2)^{1/2}$</td>
<td>$W = k_w (n_r^2 - n_s^2)^{1/2}$</td>
</tr>
</tbody>
</table>

$V = k_f (n_r^2 - n_s^2)^{1/2}$

$W = k_w (n_r^2 - n_s^2)^{1/2}$

IV. Conclusions

We have shown that an asymmetric slab waveguide can be characterized by two normalized guide parameters. The first is the familiar parameter $V$ that contains the film thickness and the optical frequency. The other is an asymmetry measure that is defined differently for the TE and TM modes. These two parameters determine the cutoff and dispersion characteristics of the guide as well as its effective thickness. A summary of the definitions of the basic waveguide parameters is given in Table II. The introduction of these parameters allows us to plot two universal charts, from which we can determine the guide index and the effective thickness for any slab waveguide configuration by following simple scaling rules. These two charts are applicable to TE modes without restriction and also directly to TM modes with the restriction to small film-substrate index differences.

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References